11/29/21 Lecture Notes Section 16.7
Surface Integrals

Last Time: $\iint_S f(x,y,z)\,dS = \iint_D f(x(u,v),y(u,v))\,|\vec{S}_u \times \vec{S}_v|\,dA$

where $\vec{S}(u,v)$ parameterizes the surface S on domain D

Ex: Compute $\iint_S x^2\,dS$ for S the surface of the unit sphere centered at the origin

$\vec{S}(\theta,\varphi) = \langle \sin(\varphi)\cos(\theta), \sin(\varphi)\sin(\theta), \cos(\varphi)\rangle$ ← spherical coords of unit sphere

$(\theta,\varphi) \in [0,2\pi] \times [0,\pi]$

$\vec{S}_\theta = \langle -\sin(\varphi)\sin(\theta), \sin\varphi\cos(\theta), 0\rangle$

$\vec{S}_\varphi = \langle \cos(\varphi)\cos(\theta), \cos(\varphi)\sin(\theta), -\sin(\varphi)\rangle$

$$\vec{S}_\theta \times \vec{S}_\varphi = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\sin(\varphi)\sin(\theta) & \sin(\varphi)\cos(\theta) & 0 \\ \cos(\varphi)\cos(\theta) & \cos(\varphi)\sin(\theta) & -\sin(\varphi) \end{vmatrix}$$

$= \langle -\sin^2(\varphi)\cos(\theta), -(\sin^2(\varphi)\sin(\theta), -\sin(\theta)\cos(\theta)\sin^2(\theta) - \sin(\varphi)\cos(\varphi)\cos^2(\theta)\rangle$

$-\sin\varphi \langle \sin(\varphi)\cos(\theta)\ \sin(\varphi)\sin(\theta), \cos(\varphi)\rangle$

$$|\vec{S}_\theta \times \vec{S}_\varphi| = \sin\varphi \sqrt{\sin^2(\varphi)\cos^2(\theta) + \sin^2(\varphi)\sin^2(\theta) + \cos^2(\varphi)}$$

$$= \sin\varphi \cdot 1$$

$$\iint_S x^2 = \int_{\theta=0}^{2\pi}\int_{\varphi=0}^{\pi} \sin^2(\varphi)\cos^2(\theta)\sin(\varphi)\,d\varphi\,d\theta$$

$$= \int_{\theta=0}^{2\pi}\cos^2(\theta)\,d\theta \cdot \int_{\varphi=0}^{\pi}\sin^3(\varphi)\,d\varphi \qquad \text{fubini}$$

$$= \frac{1}{2}\int_{\theta=0}^{2\pi}(1+\cos(2\theta))\,d\theta \cdot \int_{\varphi=0}^{\pi}\sin\varphi(1-\cos^2(\varphi))\,d\varphi$$

$$= \frac{1}{2}\left[\theta + \frac{1}{2}\sin(2\theta)\right]_{\theta=0}^{2\pi} \int_{\varphi=0}^{\pi} -(1-u^2)\,du$$

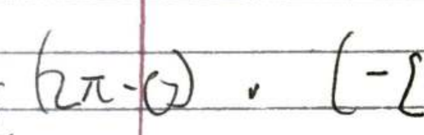
$u = \cos\varphi$
$du = -\sin\varphi\,d\varphi$

$$= \frac{1}{2}(2\pi - 0) \cdot \left(-\left[u - \frac{1}{3}u^3\right]_{\varphi=0}^{\pi}\right)$$

$$\pi \cdot \left(\left(-(-1) - \frac{1}{3}(-1)\right) - \left(1 - \frac{1}{3}1\right)\right) = \frac{4\pi}{3}$$

Want A theory of surface integrals of vector fields... First need to understand what "orientation" means for surfaces...

think back to line integrals:



changing orientation negates integrals

Orientation ≈ choice of direction...
RHR

Orientation should be controled by the normal vector of the tangent plane to the surface at a given point

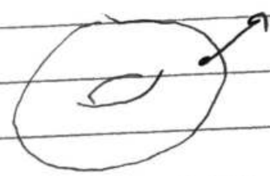
Positive orientation determined by right hand rule...

$\vec{S}_u \times \vec{S}_v$ should point "out" or "up" for positive orientation

Ex:  pointing outward = positive orientation

 pointing inward = negative orientation.

Can we always choose a consistent orientation for a surface

 ~~torus w/ outward~~ normal...

Möbius Band or Moebius Band → cylinder w/ half twist

Click on link in website

Surface is non orientable → no consistant choice of normal

Vid "Mui I M bius stri"

NB: Our Surface integral from now on will assume an orientable surface...

i.e. $\vec{n}(u,v) = \dfrac{\vec{S}_u \times \vec{S}_v}{|\vec{S}_u \times \vec{S}_v|}$ for parameterization $\vec{S}(u,v)$ is consistent...

↳ E.g. the Möbus band is excluded!

Def^n: Given a vector field $\vec{V}$ on $\mathbb{R}^3$ and an orientable surface S w/ parameterization $\vec{S}(u,v)$, the flux of $\vec{V}$ across S is

$$\iint_S \vec{V} \cdot d\vec{S} = \iint_S \vec{V} \cdot \vec{n}(u,v)\, dS$$

$$= \iint_S \vec{V} \cdot \frac{\vec{S}_u \times \vec{S}_v}{|\vec{S}_u \times \vec{S}_v|}\, dS = \iint_D \vec{V} \cdot (\vec{S}_u \times \vec{S}_v)\, dA.$$

Example: Compute flux of $\vec{v}=\langle z, y, x\rangle$ across the unit sphere centered @ the origin.

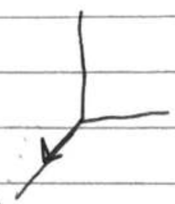
★ If no orientation is given, assume the "counter-clockwise from above" or "outward" orientation.

Sol: From earlier S is parameterized by
$\vec{S}(\theta,\varphi)=\langle \sin(\varphi)\cos(\theta), \sin(\varphi)\sin(\theta), \cos(\varphi)\rangle$

on $(\theta,\varphi)\in[0,2\pi]\times[0,\pi]$ and has

$S_\theta \times S_\varphi = -\sin\varphi\langle \sin(\varphi)\cos(\theta), \sin(\varphi)\sin(\theta), \cos(\varphi)\rangle$

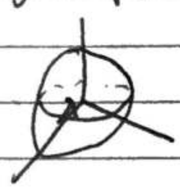
Q: is that the outward normal?
A: Check the "east pole" $(1,0,0)$



$(\theta,\varphi)=(0,\frac{\pi}{2})$

$(\vec{S}_\theta\times\vec{S}_\varphi)(0,\frac{\pi}{2}) = -1\langle 1,0,0\rangle = \langle -1,0,0\rangle$

inward orientation



$\therefore$ We must work w/ $-\vec{S}_\theta\times\vec{S}_\varphi$ instead

$\therefore$ The flux of $\vec{v}$ across S is

$$\iint_S \vec{v} \cdot d\vec{s} = \iint_D \vec{v}(u,v) \cdot (-\vec{S}_u \times \vec{S}_v)\, dA$$

$$= \iint \langle \cos(\varphi), \sin(\varphi)\sin(\theta), \sin(\varphi)\cos(\theta) \rangle \cdot \sin(\varphi)\langle \sin(\varphi)\cos(\theta), \sin(\varphi)\sin(\theta), \cos(\varphi) \rangle$$

$$= \iint_D \sin(\varphi)\left(2\cos(\varphi)\sin(\varphi)\cos(\theta) + \sin^2(\varphi)\sin^2(\theta)\right) dA$$

$$= 2\iint_D \cos(\varphi)\sin^2(\varphi)\cos(\theta)\, dA + \iint_D \sin^3(\varphi)\sin^2(\theta)\, dA$$

$$= 2\int_{\varphi=0}^{\pi}\int_{\theta=0}^{2\pi} \cos(\varphi)\sin^2(\varphi)\cos(\theta)\, d\theta\, d\varphi$$

$$= 2\int_{\varphi=0}^{\pi} \cos(\varphi)\sin^2(\varphi)\left[\sin\theta\right]_{\theta=0}^{2\pi} d\varphi$$

$$2\int_{\varphi=0}^{\pi} 0\, d\varphi = 0, \quad \int_{\varphi=0}^{\pi}\int_{\theta=0}^{2\pi} \sin^3(\varphi)\cdot\frac{1}{2}(1-\cos(2\theta))\, d\theta\, d\varphi$$

$$= \int_{\varphi=0}^{\pi} \frac{1}{2}\sin^3(\varphi)\left[\theta - \frac{1}{2}\sin(2\theta)\right]_{\theta=0}^{2\pi} d\theta$$

$$= \int_{\varphi=0}^{\pi} \frac{1}{2}(2\pi - 0)\sin(\varphi)(1-\cos^2(\varphi))\, d\varphi \qquad \begin{matrix} u = \cos(\varphi) \\ du = -\sin\varphi\, d\varphi \end{matrix}$$

$u(\pi) = -1 \qquad u(0) = 1$

$$= \pi\int_{u=1}^{-1} -1(1-u^2)\, du = -\pi\left[u - \frac{1}{3}u^3\right]_{u=1}^{-1}$$

$$= \boxed{\frac{4\pi}{3}} \Rightarrow 0 + \frac{4\pi}{3} = \boxed{\frac{4\pi}{3}}$$

Exercise: Compute $\iint \vec{v} \cdot d\vec{S}$ for

$\vec{V} = \langle y, x, z \rangle$ on the boundary of the solid enclosed by the paraboloid $z = 1 - x^2 - y^2$ and the plane $z = 0$

1. check orientation of both. bottom piece will point downward.